Self-organised Criticality in Complex Networks

Radu Dobrescu*, Mihai Tanase*

*Politehnica University of Bucharest, Faculty of Control and Computers,
Spl. Independentei 313, sector 6, Bucharest, ROMANIA (e-mail: radud@yahoo.com)

Abstract: We investigated the avalanche dynamics of the sandpile model (first presented by Bak, Tang and Wiesenfeld) on complex networks with general degree distributions. We proposed a cellular automaton based model associated to an extended set of critical exponents, estimated to characterize the avalanche properties at the nonequilibrium steady state of the model. Then we developed a similar model for networks having the threshold height of each node given as its connectivity degree. In this model, self-organized criticality emerges such that the avalanche size and the duration distribution follow power laws. The probability distribution functions are found to obey usual finite size scaling supported by negative time autocorrelation between the toppling waves. The model exhibits characteristics of both deterministic and stochastic sandpile models. The analytic solutions are supported by our numerical simulation results.

Keywords: cellular automaton, sandpile model, avalanche dynamics, power laws, scale free networks, critical exponents.

1. INTRODUCTION

The concept of self-organized criticality was proposed by Bak, Tang, and Wiesenfeld (1987) to explain why spatial fractal and fractal time series, known as “1/f fluctuations”, are so ubiquitous in nature. Power laws, scaling laws, 1/f fluctuations, or Zipf’s law refer to the nonrandom variation of a given event over space or time. The intensity of earthquakes in a certain geographic region, for example, vary along time with stronger earthquakes (i.e. those releasing greatest amounts of energy) being less frequent than the weak ones.

When a temporal series of seismic activity is ordered according their ranking, the relationship between the rank of an occurrence (n) and its frequency (f) follows a power law like:

\[ f = a n^{-\gamma} \] (1)

where a and \( \gamma \) are empirical constants. In addition to the variation in the magnitude of earthquakes, power laws have been reported in the description of several phenomena like variations on stock prices, large and small biological extinction events or intensity of pulsar emissions (Bak, 1994; Correll, 2008).

Therefore, Bak et al. (1987) proposed that the most basic physical system exhibiting this phenomenon is constituted by a sand pile when newgrains are added. Sand piles have a conic form with a defined slope. Once that critical slope is reached, the addition of more sand starts to produce avalanches that reduce the height of the pile. If the slope is greater than the critical value, an unstable situation results, in which avalanches of sand occur very readily and reduce the slope until it reverts to the critical value. Thus the sand piles are naturally “attracted” to the critical value of the slope, without any special external adjustment being necessary (hence “self-organized criticality”). In the critically self-organized state, after the addition of a single grain we can observe small avalanches, big avalanches, or no movement at all. In this state, it is impossible to predict what will happen to the system after each single grain addition. Although one cannot predict the consequence of adding a single grain, the behavior of the entire pile can be completely described in statistical terms. Small avalanches are more frequent than big ones and the relative frequency among them follows an exponential relationship, a power law. According to Jensen (1998): “The systems are complex in the sense that no single characteristic even size exists: there is not just one time and one length scale that controls the temporal evolution of these systems. Although the dynamical response of the systems is complex, the simplifying aspect is that the statistical properties are described by simple power laws”.

Another characteristic of systems presenting self-organized criticality is the presence of selfsimilarity. An object is considered self-similar if it has the same aspect on any scale, i.e., if it is a geometric object made up of smaller copies of itself (Serrano et al., 2008). The lack of an absolute scale and self-similar appearance can also be found in certain geometric objects, such as fractals. The identification of power laws and SOC in a given system, however, is not self evident. There are no consensual mathematical procedures to detect its presence in experimental data. Even the simplest physical system constituted by a sand pile can have its behavior described by alternative ways. If grains of sand are slowly dropped onto the pile at random positions, avalanche activity can be observed on the slope of the pile. If the pile is large enough, no important interference between individual avalanches takes place. In this situation, the instantaneous activity of the pile can be statically described by considering a random linear superposition of individual avalanches. The power spectrum of the total activity can then be calculated as...
the appropriately weighted sum of power spectra of the individual avalanches.

Although used to explain a diversity of phenomena, the entire idea of self-organized criticality should not be considered as a comprehensive theory at the same level of thermodynamics, for example. The idea of self-organized criticality is attractive due to its proposal of simplification and unification of general principles but its use outside physical systems requires a distinction between simple metaphor and heuristic hypothesis based in empirical evidence.

2. CELLULAR AUTOMATON MODEL FOR SOC

As it was mentioned above, one of the most remarkable features that characterizes self-organized criticality is the events’ power law distribution. This feature, combined with the abundance of real-world networks with scale-free (SF) degree distribution (Dorogovtsev and Mendes, 2003; Barabási, 2007; Serrano et al., 2008), may give rise to the suspicion that a relationship exists between the two issues. Although there are no formal procedures to detect power laws, its identification is an indispensable condition before any further theoretical speculation can be done. Power laws and spontaneous scale invariance are essential for the empirical demonstration of SOC as organizational principle in any natural or living system (Jensen, 1998).

To sustain these relation, let use the rules of the sandpile model, which is a simple, intuitive example of self-organized criticality. For this model we consider a cellular automaton (CA) whose configuration is determined by the integer variable $c_{i}$ (the “sand column’s” height) at every node $i$ in a random network with a given degree distribution $p(k)$. The dynamics are defined by the following two simple rules: 1) A grain of sand is added at a randomly selected node $i$: $c_{i} \rightarrow c_{i}+1$. 2) A sand column with a height $c_{i} \geq k_{i}$, where $k_{i}$ is equal to degree of node $i$, becomes unstable and collapses by distributing one grain of sand to each of its $k_{i}$ neighbors. This may cause some of them to become unstable and to collapse in the next time step. This in turn can lead to an avalanche of subsequent instabilities. During the evolution a small fraction $f$ of grains is lost, which prevents the system from becoming overloaded. When the avalanche dies another grain of sand is added. In the sandpile model the avalanches’ sizes’ distributions (measured as the total number of topplings in the avalanche), the avalanches’ areas (the number of distinct nodes participating in a given avalanche) and the avalanches’ durations, as well as many other statistics, follow power law distributions. Moreover, the measured distributions characteristic exponents depend on the network dimension and topology.

The one-dimensional sand pile model (1D_SP_CA) can be exactly solved and yields an avalanche area distribution $P_{a}(a,L)=a^{-\tau}F_{a}(a/L)$ where $a$ is an avalanche’s area (total number of lattice sites toppling at least once during the avalanche), $L$ is the system’s size and $F_{a}(a/L)$ is a finite size scaling function (Fronczak, 2006). If we consider a current cell $h_{i}$, then the first general rule means that adding a grain $h_{i} \rightarrow h_{i}+1$ and $z_{i} \rightarrow h_{i}+1$ is the resulted slope. The distribution of a grain to the neighbors change the slope is $z_{i} \rightarrow z_{i}{\rightarrow} z_{i}+1$ and $z_{i} \rightarrow z_{i}+1$. The distribution via $h_{i} \rightarrow h_{i}-1$ and $h_{i} \rightarrow h_{i}+1$ leads to $z_{i} \rightarrow z_{i}{\rightarrow} z_{i}+1$ and $z_{i} \rightarrow z_{i}+1$ (see fig.1).

![Fig.1. Slope changes in the 1D_SP_CA](Image 315x457 to 544x677)

Despite plenty of numerical studies of the twodimensional (2D_CA) model and its analytical tractability, the scaling behavior in this case is probably not yet completely understood (Stella and De Menech, 2001) partly due to the fact that avalanches involve a large fraction of multiple topplings. In our model, we consider that grains of sand are added to a grid, causing the slope to increase: $z(i,j) \rightarrow z(i,j) +1$. If a threshold $z > z_{c}$ is reached, the grains are redistributed after the following rules: $z(i,j) \rightarrow z(i,j)-4$; $z(i\pm 1,j) \rightarrow z(i\pm 1,j)+1$; $z(i,j\pm 1) \rightarrow z(i,j\pm 1)+1$. It follows that an avalanche’s area $a$ differs from an avalanche’s size $s$. A result is a finite size scaling for $P_{a}(a,L)$ with the characteristic exponent $\tau_{s}=6/5$, while the distribution $P_{a}(s,L)$ exhibits multifractal scaling behavior. Goh et al. (2003) have studied sandpile dynamics on scale free networks $p(k) \sim k^{-\gamma}$, where the degree $k$ of a vertex is the number of links incident on the vertex and they have shown that the avalanche’s area exponent $\alpha$ is independent of the average network connectivity $\langle k \rangle$ and changes with the degree distribution’s exponent $\gamma$. They have obtained $\alpha=2\gamma$ in the range $2<\gamma<3$ and $\alpha=1.5$ for $\gamma>3$. The question we discuss is how does the avalanche’s area exponent behave when the network’s topology depends on a power law for connectivity.

3. AVALANCHE DYNAMICS OF SCALE FREE NETWORKS

Many complex systems can be viewed as a network made of nodes and links. A node represents a constituents of the
system while a link an interaction between them. What interests us recently is that the degree distributions of complex networks such as the Internet and biological networks follow a power law (Dobrescu and Purcarea, 2009). The networks following such a power-law degree distribution are called scale-free networks (SFN). Such a slowly-decaying degree distribution implies the presence of a non-negligible fraction of hubs, i.e., nodes of very large degrees, which would exist with exponentially small probabilities in networks with a Poisson degree distribution or an exponential one. In this work, we investigate the effect of network topology represented by the degree distribution on the avalanche dynamics in the sandpile model, which was studied extensively on regular lattices as a prototypical system showing self organized criticality (SOC). In Euclidean space, the main SOC feature of the model is the emergence of a power law in the avalanche size distribution. Only few studies have been performed on the sandpile model on complex networks with general degree distributions including power-law ones, but the common approach is that the network reorganizes its structure as a consequence of avalanches of rewiring processes. The only parameter of the model that influences the rewiring, and consequently the network’s structure, is a type of probability in which a chosen node becomes unstable and has to be rewired. Choosing this probability as a power law sets the system in a critical state and forces the network to take a power law degree distribution.

In the present paper, instead of forcing the network to stay in a critical state, we allow the system to naturally evolve toward the critical region. In our model the considered networks’ degree distribution change due to the sandpile’s avalanches’ distribution on this network and on the other hand the avalanches’ size distribution changes because the network structure evolves. These two mechanisms influence each other and lead to the equilibrium point at which the avalanches’ distribution and the degree distribution shapes become very similar.

In order to complete the model rules, apart from the sandpile model rules presented above, we define the rewiring process in the following way: each end of a link has been assigned a value specifying the last time when it was rewired. After an avalanche of area A, the number of A “oldest” ends of links (from the whole network) are rewired to the node which triggered the avalanche. An example of a rewiring process is shown in Fig. 2.

The gray colored node in diagram A started an avalanche of size 2. The number close to each end of the link describes a moment of its last rewiring. A node can become unstable by losing links, i.e., \( c_i \to k_i \) because \( k_i \) has been decreased. Such a node will participate in the next avalanche caused by the addition of a new grain. If a node loses all its links it has no possibility of creating an avalanche or of connecting to other nodes, but this would mean to let this node redistribute 0 grains indefinitely. To avoid this problem we assume \( c_i = 1 \) for this node critical height. This means that if a grain is added to such a node, it will generate an avalanche of size 1, which will lead to \( k_i \to k_i + 1 \).

Let now consider the avalanche dynamics on networks with degree distribution \( p_k \) as follows: 1) At each time step, a grain is added at a randomly chosen node \( i \). 2) If the height at the node \( i \) reaches or exceeds a prescribed threshold \( z_i \), where we set \( z_i = k_i \), then the degree of the node \( i \), it becomes unstable and all the grains at the node topple to its adjacent nodes: \( h_j \to h_j - k_i \), and \( h_j = h_j + 1 \), where \( j \) is a neighbor node of the node \( i \). 3) If this toppling causes any of the adjacent nodes to be unstable, subsequent topplings follow on those nodes in parallel until there is no unstable node left.

The three steps can be repeated and this process defines an avalanche. As characteristic parameters we decide choose the avalanche area \( A \) (the number of distinct nodes participating in a given avalanche, the avalanche size \( S \) (the number of toppling events in a given avalanche), and the duration \( T \) of a given avalanche. Analytic solutions for the avalanche size and duration distributions can be obtained by applying the theory of multiplicative branching processes [12]. To each avalanche, one can draw a corresponding tree structure (see Fig. 3). The node where the avalanche is triggered is the originator of the tree and the branches out of that node correspond to topplings to the neighbors of that node. As the avalanche proceeds, the tree grows. The number of branches of each node is not uniform, but is equal to its own degree. The branching process ends when no further avalanche proceeds. In the tree structure, a child node born at time \( t \) is located away from the originator by a distance \( d \) along the shortest pathway. In the branching process, it is assumed that branchings from different parent nodes occur independently.

Then one can derive the statistics of tree size and lifetime analytically, which can be considered as that of avalanche size and duration since the avalanche duration \( T \) is equal to the lifetime of the tree minus one, and the avalanche size \( S \) differs from the tree size only by the number of boundary
nodes of the tree, which is relatively small when the overall tree size is very large. The avalanches in complex networks usually do not form a loop but have tree-structures. Because the probability distributions of the two quantities \( \mathcal{A} \) and \( \mathcal{S} \) behave in a similar fashion, we shall not distinguish \( \mathcal{A} \) and \( \mathcal{S} \), and use \( s \) to represent either \( \mathcal{A} \) or \( \mathcal{S} \).

The branching probability \( q(k) \) that a certain node generates \( k \) branches in the corresponding tree consists of two factors: One is the probability \( q_1(k) \) that the node receiving a grain from one of its neighbors has \( k \) degrees and the other is the probability \( q_2(k) \) that toppling indeed occurs at the node. The probability \( q_1(k) \) is equal to the degree distribution of the node at one end of a randomly chosen edge, and \( q_2(k) \) corresponds to the probability that the node has height \( k-1 \) at the moment of gaining the grain from one of its neighbors. If one assumes that there is no typical height of a node in the inactive state, regardless of its degree \( k \), then \( q_2(k) = 1/k \).

Fig. 3. Tree model for an avalanche

In fig.3(a) an avalanche triggered at the node 0 (circle) propagates to the nodes 1; 2; 3; and 4 (squares), and then to 5; 6; and 7 (hexagons). No toppling event occurs at the nodes 1; 3; 5; 6; or 7. In fig.3(b) the corresponding tree of the avalanche can be drawn with three generations \( t = 0; 1; 2 \) depending on the distance from the node 0, the originator. The avalanche size is 3, contributed by the nodes, 0; 2; 4, denoted by filled symbols.

4. SIMULATION RESULTS

The nonequilibrium steady state is defined by the constant average height of the sandpile at which the current of influx of sand grain to the system is equal to the current of outflux of the same at the open boundary. In order to characterize the physical properties of the avalanches, which occurred at the nonequilibrium steady state, simulations have been performed on the square lattice of sizes \( L=128 \) to \( L=2048 \) in multiples of 2. The first 106 avalanches were skipped to achieve the steady state. Extensive data collection has been made for each lattice size for averaging, ranging from \( 32 \times 10^6 \) avalanches for \( L=128 \) down to \( 2 \times 10^6 \) avalanches for \( L=2048 \) in ten configurations. In each configuration, the initial 10^5 avalanches are neglected again on the steady state before collecting data.

Fig. 4. Plot of average height \( \langle h \rangle \)

The average height \( \langle h \rangle \) is plotted against the number of avalanches up to \( 10^6 \) in Fig. 4 for the system size \( L=2048 \). It can be seen that a constant average height is achieved and it remains constant over a large number of avalanches. For smaller lattice sizes the steady states are reached by a smaller number of avalanches. A slight variation of the average height with the system size \( L \) is observed.

A comparison of the morphology of avalanche clusters in the steady state was also made. Typical large avalanche clusters obtained in the respective steady states are shown in Fig. 5. The avalanche clusters are generated on a square lattice of size 64x64 dropping sand grains one at a time at the center of the lattice. The clusters shown here have 21 maximum numbers of toppling in each and it is represented by the red color. Different colors correspond to different numbers of toppling of sites in an avalanche as black for 21, blue for 20-17, red for 16-13, green for 12-9, yellow for 8-5, and gray for 4-1 toppling numbers. White spaces inside the avalanche correspond to the sites that did not topple at all during the avalanche.

Fig. 5. Typical avalanches generated at the steady state

In order to characterize different physical properties of the avalanches, which occurred at the steady state, different
quantities like toppling size $s$ of an avalanche, avalanche area $a$, lifetime $t$, and spatial extension $l$ are measured. Toppling size $s$ is defined as the total number of toppling which occurred in an avalanche. Avalanche area $a$ is equal to the number of distinct sites toppled in an avalanche. Lifetime $t$ of an avalanche is taken as the number of parallel updates to make all the sites undercritical. Spatial extension $l$ of an avalanche is given by:

$$l = \sqrt{\frac{2}{a} \sum_{i=1}^{a} (r_i - \bar{r})^2} / a,$$

where $\bar{r}_0 = \sum_{i=1}^{a} r_i / a$ is the position vector of the distinct sites toppled (Sandra et al., 2007). The related critical exponents are estimated determining the probability distributions of all these properties ($s$, $a$, $t$, and $l$). The probability distribution function of an avalanche-related quantity $x$ at the steady state of a given system size $L$ is expected to obey power law behavior given by:

$$P(x, L) \sim x^{-\tau_x} f(x/L^D_x),$$

where $\tau_x$ is the corresponding critical exponent and $x$ stands for $s$, $a$, $t$, and $l$. $f(x/L^D_x)$ is the finite size scaling function and $D_x$ is called a capacity dimension. In the $L \to \infty$ limit, the scaling function $f(0)$ becomes a constant and the power law behavior can be approximated as $P(x) \sim x^{-\tau_x}$. The corresponding exponents $\tau_x$ can be estimated from the slope of the best fitted straight line through the data points in logarithmic scale. Data are collected in bins of intervals of 10s, 100s, 1000s, and so on. Finally, the data are normalized by the bin widths. In order to extract the critical exponents related to other avalanche properties the same procedure has been followed. The values of the critical exponents are:

$$\tau_s = 1.29 \pm 0.01; \quad \tau_a = 1.33 \pm 0.01; \quad \tau_t = 1.48 \pm 0.01; \quad \tau_l = 1.66 \pm 0.01,$$

almost the same as those presented by Lübeck and Usadel (1997).

5. CONCLUDING REMARKS

To conclude, in this paper we have presented by numerical simulations how Bak-Tang-Wiesenfeld’s sandpile model (BTW) avalanche dynamics and the complex network structure may influence each other. Such an interplay between dynamics and structure leads to self-organization in which the avalanches distribution and degree distribution shapes become similar.

As common properties for self-organized criticality we considered a constant energy input, definite thresholds exhibiting the ability to store energy and the local interactions. The characteristics shown as common features were avalanches at all scales (with critical exponents for size and for time) and power-law frequency-size distributions. We suspect that the value of both exponents may be universal for a large class of SOC phenomena in which the critical behavior occurs not “on” the network structure but “in” the structure. A new two dimensional cellular automaton “quasideterministic” sandpile model was defined in order to study the effect of local external bias on self-organized critical systems. The model has microscopic properties such as mass conservation, open boundary, and local determinism in sand grain distribution on toppling as that of BTW. Calculating an extended set of critical exponents it is found that some of the exponents are close to that of BTW.

To check the analytic solutions, we performed numerical simulations for SF networks in the static model, since nontrivial values of exponents are expected to be observed in networks with power-law degree distributions. Additionally, we allow a small fraction of grains to be lost in the simulations, to prevent the system from being overloaded in the end. The numerical values for different critical exponents are presented. One can affirm that the BTW sandpile model can be applied on complex networks with general degree distributions. To account for the heterogeneity of the system, the threshold height of each vertex is set to be equal to the degree of the vertex. By mapping to the multiplicative branching process, we can obtain the asymptotic behaviors of the avalanche size and duration distributions analytically. The fact that the duration exponent $\tau$ increases as the degree exponent $\gamma$ decreases implies that the hubs sustain large numbers of grains, explaining the resilience of the network under avalanche phenomena. This is reminiscent of the extreme resilience of the network under random removal of vertices for $\gamma>3$.

REFERENCES


